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A new approach to inversion of surface wave dispersion relation for determination of depth distribution of non-uniform stresses in elastic materials

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Abstract

Ultrasonic surface (Rayleigh) waves become dispersive when propagating on non-uniformly stressed media. In light of this, the Acoustoelastic effect on their propagation in deformed but initially isotropic materials has been investigated in the past, in order to determine the surface stress and gradients of stress with depth. An energy perturbation approach considerably reduces the complexity in the treatment of the Acoustoelastic effect and inversion of the perturbation relation offers an advantageous route to obtaining the stress gradients. This paper presents a new mechanism for effecting this inversion, which tries to overcome the effects of the ill-posed nature of the problem. Preliminary simulation results for commonly occurring stress profiles are presented.
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1. Introduction

Stress states existing in materials without the presence of any external cause are called residual stresses (James and Lu, 1996). Many common manufacturing processes induce residual stresses in materials and machine components. The performance of materials under different operating conditions depends on the residual stress present and their presence can have either a beneficial or harmful effect. In certain cases, like

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that of presence of a tensile surface stress, they can lead to catastrophic failure by adding to the loads to be borne by parts and accelerating crack growth. On the other hand, compressive residual stresses (introduced sometimes deliberately by operations like shot peening) can impede crack growth and greatly enhance the life of components (Noyan and Cohen, 1991). Hence the measurement of residual stresses is of crucial importance for stress analysis and component design.

In practical conditions, ultrasonic methods are attractive for non-destructive characterization of residual stresses, because they can provide information pertaining to the interior of the material, and can be applied to a wide range of materials (Lindgren et al., 1993). Also, they allow instrumentation that is convenient, portable, inexpensive and free from radiation hazards (Thompson et al., 1996).

Ultrasonic surface waves can be useful when only one side of the component to be investigated is accessible and when we require stress profile information. In this paper, we examine the possibility of using dispersion of Rayleigh surface waves for such a configuration, to obtain stress as a function of depth. Rayleigh waves occur at the surface of a semi-infinite medium, for example, the free surface of a thick plate (a thick plate is one in which the plate thickness is very large in comparison to the wavelength) (Rose, 1999). Since they travel along the surface of a sample, Rayleigh waves can also be used to inspect curved surfaces and difficult geometries, which cannot be probed otherwise.

Ultrasonic methods for stress measurement commonly rely upon the effect of pre-stress on the propagation velocity or phase of ultrasonic waves, called the 'Acoustoelastic effect'. Traditional treatment of the Acoustoelastic effect starts with a form of Naviers' displacement equations of motion, containing terms for applied initial stress. This theory was developed by Hughes and Kelly (1953), based on the theory of finite deformations by Murnaghan (1951). This approach was first used to analyze the Acoustoelastic effect on Rayleigh waves by Hayes and Rivlin (1961), and extended by Iwashimizu and Kobori (1978) to the general case in which the propagation direction does not coincide with one of the principal axes of strain. It leads to a linear relationship (generally) between velocity change and applied stress. These works restricted themselves to the case of presence of uniform stress fields. Though Duqennoy et al. (1999) have used a similar approach to obtain arbitrary stress profiles from measured velocity profiles, their method is restricted to cases where the depth direction of the sample is accessible and stress is non-uniform only in that direction. It is difficult to extend such an analysis of the Acoustoelastic effect to the general case of an arbitrary in-homogenous beam passing through an in-homogenously stressed medium, as the calculations can get quite complicated.

An alternative approach is to apply perturbation theory to predict the effect. Perturbation theory is concerned with small changes in the solution, caused by small changes in the physical parameters of the problem (Nayfeh, 1983). It serves as a powerful tool which provides analytical approximations to solve problems not readily attacked by direct computation (Norris and Sinha, 1995; DiPerna and Feit, 2000; Willatzen, 2001). Auld (1990) first developed a perturbation formula for the elastic surface wave case. This was applied by Tittman and Thompson (1973) to the dispersion problem and was further studied by Szabo (1975). Richardson (1977) and Richardson and Tittman (1977) based on the work by Tittman and Thompson (1973), looked at the inverse problem of obtaining material property gradients from surface wave dispersion. They sought to address the ill-posedness of the inverse problem by an Estimation Theory based approach. Hirao et al. (1981) first analyzed the case of Acoustoelasticity of Rayleigh wave for the presence of non-uniform stress state, by taking account of high order perturbations of the wave equation itself. They provided theoretical and experimental confirmation of the anticipation that Rayleigh wave Acoustoelasticity gets dispersive (that is, frequency dependent) for such cases. This approach was further extended and generalized by Kline and Jiang (1996). Husson and Kino (1982) took a different approach to the application of perturbation theory to the characterization of Acoustoelastic effect. This method is based on a Lagrangian description of the motion of particles and the use of energy perturbation methods, in which the application of stress to a medium is regarded as a perturbation of the medium. Based on this work, later, Husson (1985) derived an integral equation relating the change of phase of a Rayleigh wave and the applied stress.

Ditri and Hongerholt (1996) and Ditri (1997) sought to unify the conclusions of Hirao et al. (1981) and the results of Husson (1985) to examine the possibility of using Rayleigh wave dispersion to obtain stress profile in initially isotropic materials. Ditri (1997) suggested that, the problem of obtaining the stress distribution from measured values of change of phase of ultrasonic Rayleigh waves propagating in a stressed medium, constitutes the 'Inverse problem'. We present a new approach to achieve this inversion that addresses the ill-posedness of the problem.

2. The forward problem

The forward model is based on the work by Husson (1985) and corrections published by Ditri and Hongerholt (1996). We follow a notation similar to the one used by Ditri (1997).

A Rayleigh wave, upon propagation over a certain distance on the surface of the material, undergoes a phase shift $\phi(\omega)$. Hence an initial particle velocity v becomes $ve^{(i\phi)}$. Let $\phi^0(\omega)$ denote the phase shift which would have been experienced by the wave propagating on an equivalent stress free medium and $\delta\phi(\omega)$ is the phase difference $(\phi - \phi^0)$. For Rayleigh waves propagating along the a_3 direction on the free surface $a_1 - a_3$ (see Fig. 1) of an initially isotropic elastic medium, the phase difference can be expressed as:

$$\delta\phi(\omega) = -\frac{\omega}{4P}\int_V G(a_2,\omega)\,\mathrm{d}V \tag{2.1}$$

where ω denotes the circular frequency, P denotes the power flow, or the average power carried by the Rayleigh wave over one time period, per unit width in a direction perpendicular to the travel direction and V, a volume enclosing the Rayleigh wave with fronts extended infinitely in the direction perpendicular to propagation direction. 'a_2' denotes the depth direction. The detailed expression for P is given in Appendix A. It is worthy to note that P is a function of density, phase velocities of the Rayleigh, transverse and longitudinal waves in the unstressed medium and ω. Also, P/ω is a constant.

From this step, a slightly different notation than Ditri (1997) is introduced, in order to render the expressions more compact. If b_i, $i \in \{1,2,3\}$ are the components of the initial static displacements of the medium (due to the applied pre-stress) and a_i, $i \in \{1,2,3\}$ are the coordinates of a material particle in the unstressed state, (thus, $\frac{\partial b}{\partial a}$'s form the initial deformation gradients in the medium), λ, μ are the second order (Lame), and l, m, n the third order (Murnaghan) elastic constants of the medium, the integrand $G(a_2,\omega)$ is given by

$$G = \frac{\partial b_m}{\partial a_m}\psi_1(a_2,\omega) + \frac{\partial b_2}{\partial a_2}\psi_2(a_2,\omega) + \frac{\partial b_3}{\partial a_3}\psi_3(a_2,\omega) - \frac{\partial b_1}{\partial a_1}\psi_4(a_2,\omega) \tag{2.2}$$

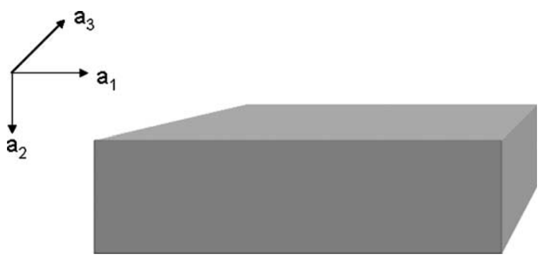

Fig. 1. The coordinate system.

where

$$\begin{aligned}
\psi_1(a_2,\omega) &= \{(2l+\lambda)[F_1(a_2,\omega)+F_2(a_2,\omega)+F_3(a_2,\omega)]+(\lambda+m)F_4(a_2,\omega)+mF_5(a_2,\omega)\}\\
\psi_2(a_2,\omega) &= \{(2\lambda+6\mu+4m)F_1(a_2,\omega)+\mu[2F_4(a_2,\omega)+F_5(a_2,\omega)]\}\\
\psi_3(a_2,\omega) &= \{(2\lambda+6\mu+4m)F_2(a_2,\omega)+\mu[2F_4(a_2,\omega)+F_5(a_2,\omega)]\}\\
\psi_4(a_2,\omega) &= \{(\lambda+2m-n)F_3(a_2,\omega)+(n/2)[F_4(a_2,\omega)+F_5(a_2,\omega)]\}
\end{aligned} \tag{2.3}$$

And F_i, $i\in\{1,2,3,4,5\}$ are the displacement gradients caused by the Rayleigh wave, given by:

$$F_i(a_2,\omega)=\omega^2 f_{ij}\mathrm{e}^{-2\omega k_j a_2} \tag{2.4}$$

The constants f_{ij} and k_j, $j\in\{1,2,3\}$ are defined in Appendix A (Throughout this paper, summation over repeated index is implied).

2.1. Uniaxial stress state

Eq. (2.1) has been specialized by Ditri and Hongerholt (1996) to cases of propagation of the Rayleigh wave along and perpendicular to the applied stress, using the notation $\delta\phi^{\alpha\beta}$ to represent the possibly frequency dependent change in phase, experienced by a Rayleigh wave propagating in the a_α direction caused by a uniaxial stress applied in the a_β direction. Here, only the final results are presented, albeit in a more compact form.

Case 1.
The Rayleigh wave propagates in the a_3 direction over a length L_0 and has uniform fields in the a_1 direction. A uniaxial normal stress $\sigma_{33}(a_2)$, which varies only with depth a_2 is applied along a_3 axis.

For this case, Eq. (2.1) is reduced to the form:

$$\delta\phi^{33}(\omega)=-\frac{L_0}{4Q}\int_0^\infty \omega^2\alpha_i f_{ij}\mathrm{e}^{-2\omega k_j a_2}\sigma_{33}(a_2)\,\mathrm{d}a_2 \tag{2.5}$$

where $Q=P/\omega$; α_i, which are functions of the Lame and Murnaghan constants alone, are given in Appendix A.

Case 2.
The Rayleigh wave propagates in the a_3 direction over a length L_0 and has uniform fields in the a_1 direction. A uniaxial normal stress $\sigma_{11}(a_2)$, which varies only with depth a_2 is applied along a_1 axis.

For this case, Eq. (2.1) reduces to the form:

$$\delta\phi^{31}(\omega)=-\frac{L_0}{4Q}\int_0^\infty \omega^2\beta_i f_{ij}\mathrm{e}^{-2\omega k_j a_2}\sigma_{11}(a_2)\,\mathrm{d}a_2 \tag{2.6}$$

β_i, which are functions of the Lame and Murnaghan constants alone, are given in Appendix A.

2.2. Biaxial stress state

If we have two stresses $\sigma_{11}(a_2)$ and $\sigma_{33}(a_2)$ in the medium, these two together constitute a biaxial stress state. Ditri (1997) has shown that because the change of phase is a linear functional of the applied stress, the effect of biaxial stress state is the sum of the effects of each stress individually.

The notation $\delta\phi^{\alpha(\beta+\gamma)}$ is used to represent the change in phase, experienced by a Rayleigh wave propagating in the a_α direction caused by a biaxial stress applied with principal directions along the a_β and a_γ and directions. We can therefore write:

$$\begin{aligned}\delta\phi^{3(1+3)}(\omega) &= \delta\phi^{31}(\omega) + \delta\phi^{33}(\omega)\\ \delta\phi^{1(1+3)}(\omega) &= \delta\phi^{11}(\omega) + \delta\phi^{13}(\omega)\end{aligned} \tag{2.7}$$

where $\delta\phi^{11}(\omega)$ and $\delta\phi^{13}(\omega)$ are given by Eqs. (2.5) and (2.6) and

$$\delta\phi^{11}(\omega) = -\frac{L_0}{4Q}\int_0^\infty \omega^2\alpha_i f_{ij}\mathrm{e}^{-2\omega k_j a_2}\sigma_{11}(a_2)\,\mathrm{d}a_2 \tag{2.8}$$

$$\delta\phi^{13}(\omega) = -\frac{L_0}{4Q}\int_0^\infty \omega^2\beta_i f_{ij}\mathrm{e}^{-2\omega k_j a_2}\sigma_{33}(a_2)\,\mathrm{d}a_2 \tag{2.9}$$

Thus Eq. (2.7) becomes:

$$\begin{aligned}\delta\phi^{3(1+3)}(\omega) &= -\frac{L_0}{4Q}\int_0^\infty \omega^2\beta_i f_{ij}\mathrm{e}^{-2\omega k_j a_2}\sigma_{11}(a_2)\,\mathrm{d}a_2 - \frac{L_0}{4Q}\int_0^\infty \omega^2\alpha_i f_{ij}\mathrm{e}^{-2\omega k_j a_2}\sigma_{33}(a_2)\,\mathrm{d}a_2\\ \delta\phi^{1(1+3)}(\omega) &= -\frac{L_0}{4Q}\int_0^\infty \omega^2\alpha_i f_{ij}\mathrm{e}^{-2\omega k_j a_2}\sigma_{11}(a_2)\,\mathrm{d}a_2 - \frac{L_0}{4Q}\int_0^\infty \omega^2\beta_i f_{ij}\mathrm{e}^{-2\omega k_j a_2}\sigma_{33}(a_2)\,\mathrm{d}a_2\end{aligned} \tag{2.10}$$

3. The inverse problem

3.1. Case of uniaxial stress: proposed new approach

Let us first examine the case of presence of a uniaxial stress for inversion. We can write a generic expression for Eqs. (2.6) and (2.7) as shown below:

$$\delta\phi(\omega) = -\frac{L_0}{4Q}\int_0^\infty \omega^2\gamma_i f_{ij}\mathrm{e}^{-2\omega k_j z}\sigma(z)\,\mathrm{d}z \tag{3.1}$$

We assume that the difference of phase is measurable by experimentation, and that we can fit a continuous function $\delta\phi(\omega)$ to the data. Then the problem of inversion is actually the problem of finding a solution to Eq. (3.1). We note, that Eq. (3.1) is an integral equation, where the unknown quantity of interest, $\sigma(z)$ occurs within the integral sign. Specifically, it is a linear Fredholm equation of the first kind with the non-symmetric kernel

$$K_{\mathrm{F}}^{\mathrm{I}}(\omega,z) = -\frac{L_0}{4Q}\omega^2\gamma_i f_{ij}\mathrm{e}^{-2\omega k_j z} \tag{3.2}$$

For such kernels, Fredholm equations of the first kind often tend to be ill-posed. The conditions for a problem to be well posed are that it should have a solution, which is at the same time unique and stable. We cannot, in general, guarantee these conditions for any arbitrary function $\delta\phi(\omega)$ for the kind of kernel provided by Eq. (3.2). The theory for existence and uniqueness of stable solutions for Fredholm equation of the first kind imposes restrictions on the kernel and the non-homogenous term (which may not, in general, be satisfied). Even if it is known that a solution does exist, the usual iterative methods (known extensively in case of integral equations of the second kind) to reconstruct it are not available. This is due to the absence of the solution $\sigma(z)$ outside the integral of Eq. (3.1).

These features of the Fredholm integral equation of the first kind are in contrast to say the Volterra equation of the first kind, which definitely pose lesser problems. Again, in certain common instances of physical problems, the Volterra integral equations of the first kind, permit a conversion to the corresponding equation of the second kind, in which case, the solution is all the more definite and easier to obtain.

It is known that boundary value problems associated with differential equations give rise to Fredholm integral equations and that initial value problems associated with differential equations lead to Volterra integral equations. Therefore a reformulation of our problem as an initial value problem provides the advantageous prospect of dealing with the solution of a Volterra equation. This is the basis for the procedure attempted in this work, where this conversion into a Volterra equation is sought to be achieved. Looking at the problem from this standpoint, crucial use is made of the commonly known fact that the Rayleigh waves diminish rapidly beyond a depth equaling approximately one wavelength (represented by λ). Therefore along the depth direction, 'infinity' can be taken to extend to a few wavelengths, say $\approx n\lambda$ (n being a finite number). Restricting the upper limit of integration in Eq. (3.1) to this value:

Eq. (3.1) becomes:

$$\delta\phi(\omega) = -\frac{L_0}{4Q}\int_0^{n\lambda} \omega^2 \gamma_i f_{ij} \mathrm{e}^{-2\omega k_j z} \sigma(z)\,\mathrm{d}z \tag{3.3}$$

Making a substitution $s = n\lambda$ and recognizing that, $\lambda = 2\pi c/\omega$ (where 'c' is the Rayleigh wave velocity in the medium) Eq. (3.3) becomes:

$$\delta\phi(\omega) = \delta\phi(2\pi cn/s) = F(s) = -\frac{L_0}{Q}(\pi cn)^2 \int_0^{s} \frac{\gamma_i f_{ij} \mathrm{e}^{-2(2\pi cn/s)k_j z}}{s^2} \sigma(z)\,\mathrm{d}z \tag{3.4}$$

Letting $C_{ij} = -\frac{L_0}{Q}(\pi cn)^2 f_{ij}$ and $\tilde{k}_j = (2\pi cn)k_j$ we obtain:

$$F(s) = \int_0^{s} \frac{\gamma_i C_{ij} \mathrm{e}^{-2\tilde{k}_j z/s}}{s^2} \sigma(z)\,\mathrm{d}z \tag{3.5}$$

Eq. (3.4) is a Volterra integral equation of the first kind with the unknown function $\sigma(z)$ and the (non-symmetric) kernel:

$$K_{\mathrm{V}}^{\mathrm{I}}(s,z) = \frac{\gamma_i C_{ij} \mathrm{e}^{-2\tilde{k}_j z/s}}{s^2} \tag{3.6}$$

Thus we have converted the Fredholm equation of the first kind given by Eq. (3.1) into a Volterra equation of the first kind (3.5).

Also, we observe, that

$$K_{\mathrm{V}}^{\mathrm{I}}(s,s) = \frac{\gamma_i C_{ij} \mathrm{e}^{-2\tilde{k}_j}}{s^2} \neq 0 \tag{3.7}$$

And also that the derivative $\frac{\partial}{\partial z} K_{\mathrm{V}}^{\mathrm{I}}(s,z)$ exists:

$$\frac{\partial}{\partial z} K_{\mathrm{V}}^{\mathrm{I}}(s,z) = \frac{\gamma_i C_{ij}(-2\tilde{k}_j) \mathrm{e}^{-2\tilde{k}_j z/s}}{s^3} \tag{3.8}$$

Hence, we can attempt a further conversion of this equation into a Volterra equation of the second kind (Tricomi, 1957). This is achieved by setting

$$\int_0^{s} \sigma(z)\,\mathrm{d}z = E(s) \tag{3.9}$$

And integrating Eq. (3.5) by parts

$$F(s) = \left[\frac{\gamma_i C_{ij} e^{-2\tilde{k}_j z/s}}{s^2} E(z)\right]_{z=0}^{z=s} - \int_0^s \frac{\gamma_i C_{ij}(-2\tilde{k}_j) e^{-2\tilde{k}_j z/s}}{s^3} E(z)\,dz \tag{3.10}$$

That is,

$$F(s) = \frac{G}{s^2} E(s) + \int_0^s \frac{H_{ij} e^{-2\tilde{k}_j z/s}}{s^3} E(z)\,dz \tag{3.11}$$

where

$$G = \gamma_i C_{ij} e^{-2\tilde{k}_j} \quad \text{and} \quad H_{ij} = \gamma_i C_{ij}(2\tilde{k}_j) = 2\gamma_i C_{ij}\tilde{k}_j \tag{3.12}$$

Rearranging terms in Eq. (3.11):

$$E(s) = \frac{s^2}{G} F(s) - \frac{1}{G} \int_0^s \frac{H_{ij} e^{-2\tilde{k}_j z/s}}{s} E(z)\,dz \tag{3.13}$$

Eq. (3.13) is a Volterra integral equation of the second kind, with the non-homogenous term

$$J(s) = \frac{s^2}{G} F(s) \tag{3.14}$$

And the kernel

$$K_{\mathrm{V}}^{\mathrm{II}}(s,z) = -\frac{1}{G} \frac{H_{ij} e^{-2\tilde{k}_j z/s}}{s} \tag{3.15}$$

Eq. (3.13) can now be solved by standard available analytical methods. The general conditions for existence of a unique and bounded solution of a Volterra integral equation (Jerry, 1999) of the form $u(x) = f(x) + \lambda \int_a^x K(x,\xi) u(\xi)\,d\xi$ on an interval $[a,b]$ are that the function $f(x)$ be integrable on the interval and that the kernel $K(x,\xi)$ be integrable in the triangle $a \leqslant x \leqslant b$, $a \leqslant \xi \leqslant x$.

In the case of Eq. (3.13), since we have assumed that we can obtain a continuous function $\delta\phi(\omega)$ (and hence a continuous and integrable function $F(s)$, since continuity implies integrability) on $[0,s]$.

It can be observed that the kernel $K_{\mathrm{V}}^{\mathrm{II}}(s,z) = -\frac{1}{G}\frac{H_{ij} e^{-2\tilde{k}_j z/s}}{s}$ is continuous in 'z' and 's' on any triangle, $0 \leqslant s \leqslant b$, $0 \leqslant z \leqslant s$. Therefore one can always guarantee a unique and bounded solution, for any function continuous $\delta\phi(\omega)$, and by extension, for any kind of continuous input stress function.

Often, analytical procedures to obtain the solution become cumbersome. We can then attempt numerical methods, by approximating the integral in Eq. (3.13) as a sum of terms using quadrature rules:

$$E(s) = J(s) + \sum_{q=0}^{n} K_{\mathrm{V}}^{\mathrm{II}}(s, z_q) E(z_q) w(z_q) \tag{3.16}$$

Since we use either z or s as the independent variable for the solution E, we can call $s_0 = z_0 (=0)$, $s = s_n = z_n$ (where z_n is the end point we chose for z) and $s_p = s_0 + p\Delta z = z_0 + p\Delta z$, that is, $s_p = z_p$. The value of the kernel $K_{\mathrm{V}}^{\mathrm{II}}(s_p, t_q)$ vanishes for $t_q > s_p$, as the integration ends at $t_q \leqslant s_p$ Therefore, we will have the system of $n+1$ equations (Writing: $K_{\mathrm{V}}^{\mathrm{II}}(s_p, t_q) = K$, $q \leqslant p$, $J(s_p) = J_p$, $w(z_p) = w_p$ and $E(s_p) = E_p$)

$$E_0 = J_0$$
$$E_p = J_p + \sum_{q=0}^{p} K_{pq} E_q w_q \quad (p = 1, 2, \ldots, n) \tag{3.17}$$

Rearranging terms, transferring terms involving the solution E_p to the left side of (3.17) leaving the non-homogenous part J_p on the right side, we obtain the following (lower) triangular system of equations:

$$E_0 = J_0$$
$$\sum_{q=0}^{p-1} -K_{pq} w_q E_q + (1 - K_{pp} w_p) E_p = J_p \quad (p = 1, 2, \ldots, n) \tag{3.18}$$

The set of equations (3.18) can be written in a matrix form:

$$[K]\{E\} = \{F\} \tag{3.19}$$

where K is the $(n+1)\times(n+1)$ matrix of coefficients of the system of equations (3.18), $E = (E_p)$ is the column matrix of sample solutions, and $F = (F_p)$ is the column matrix of sample values of the non-homogenous part.

Eq. (3.19) yields the function E and subsequently, we can obtain the stress function $\sigma(z)$ from Eq. (3.9). The great advantage of Volterra equations of the second kind is that such a numerical approximation results in the coefficient matrix of the linear system of equations so obtained, being a (lower) triangular matrix. This is because of the variable upper limit of integration in the Volterra equation (and therefore, the kernel $K(x,t) = 0$ for $t > x$). A system of linear equations with such a natural triangular coefficient matrix is easy to solve. This is in sharp contrast to the square system of equations which result from numerical reduction of the Fredholm integral equation.

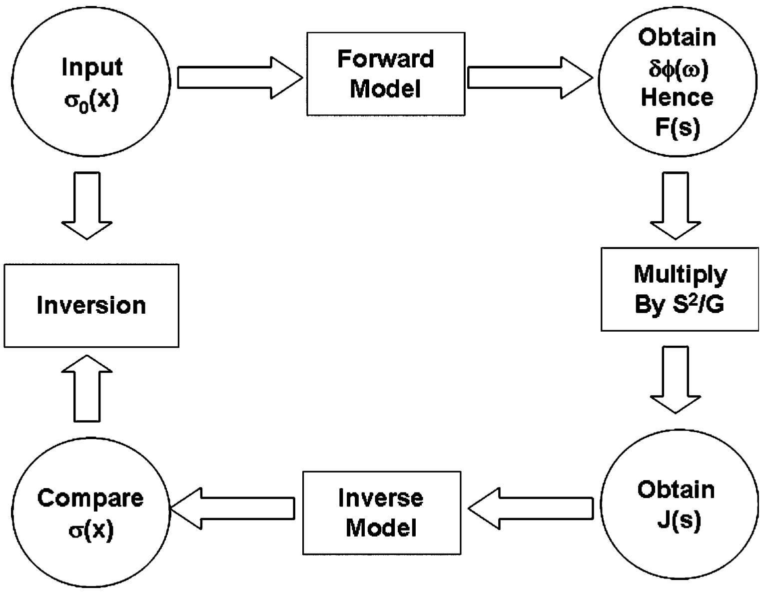

Fig. 2. Simulation methodology.

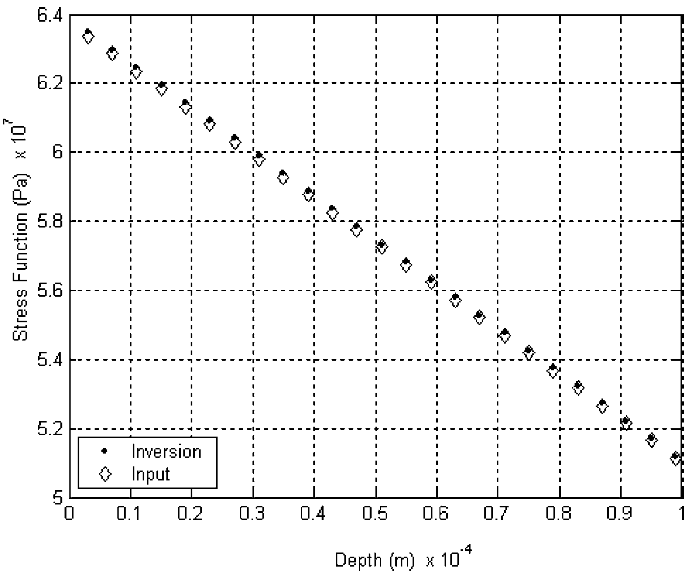

Fig. 3. Simulation result for input stress of form $\sigma = Az + B$.

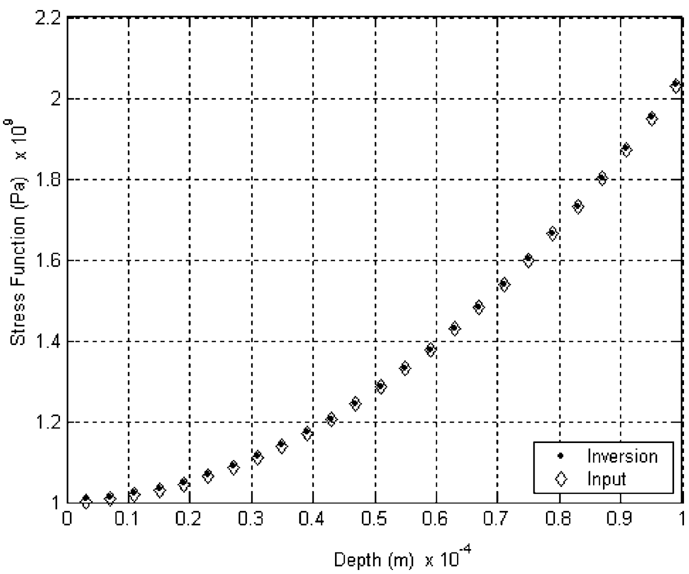

Fig. 4. Simulation result for input stress of form $\sigma = Az^2 + Bz + C$.

3.2. Simulations results

Preliminary simulation results are presented here, to demonstrate the inversion. A numerical reduction as described in Section 3.1 has been attempted, based on Simpsons's rule. The trapezoidal rule is used as a starting procedure at the second iteration. The procedure adopted for simulations is as follows (see Fig. 2): the applied pre-stress is assumed to be a known function of depth. The forward model is then used to obtain the non-homogenous term in Eq. (3.13). With this as the input, the (approximate) solution is calculated and compared with the applied stress. Fig. 3 shows the result for input stress which varies linearly with depth and Fig. 4 shows the result for a second order variation of stress with depth. It is to be noted that here, the frequency is effectively sidestepped because throughout, the calculations are in the 's' plane.

3.3. Case of biaxial stress: proposed new approach

We rewrite the set of equations (2.10):

$$\begin{aligned}\delta\phi^{3(1+3)}(\omega) &= -\frac{L_0}{4Q}\int_0^\infty \omega^2\beta_i f_{ij}\mathrm{e}^{-2\omega k_j a_2}\sigma_{11}(z)\,\mathrm{d}z - \frac{L_0}{4Q}\int_0^\infty \omega^2\alpha_i f_{ij}\mathrm{e}^{-2\omega k_j a_2}\sigma_{33}(z)\,\mathrm{d}z \\ \delta\phi^{1(1+3)}(\omega) &= -\frac{L_0}{4Q}\int_0^\infty \omega^2\alpha_i f_{ij}\mathrm{e}^{-2\omega k_j a_2}\sigma_{11}(z)\,\mathrm{d}z - \frac{L_0}{4Q}\int_0^\infty \omega^2\beta_i f_{ij}\mathrm{e}^{-2\omega k_j a_2}\sigma_{33}(z)\,\mathrm{d}z\end{aligned} \tag{3.20}$$

Assuming again that $\delta\phi^{3(1+3)}(\omega)$ and $\delta\phi^{1(1+3)}(\omega)$ are measurable and that we can fit continuous functions to them, the inverse problem now, is to solve for $\sigma_{11}(z)$ and $\sigma_{33}(z)$ from the set of equations (3.20).

Adopting the same procedure as in Section (3.1), the set of equations (3.20) can be written as:

$$\begin{aligned}F^{3(1+3)}(s) &= \frac{G_2}{s^2}E_{11}(s) + \int_0^s \frac{H_{ij}^2\mathrm{e}^{-2\tilde{k}_j z/s}}{s^3}E_{11}(z)\,\mathrm{d}z F(s) + \frac{G_1}{s^2}E_{33}(s) + \int_0^s \frac{H_{ij}^1\mathrm{e}^{-2\tilde{k}_j z/s}}{s^3}E_{33}(z)\,\mathrm{d}z \\ F^{1(1+3)}(s) &= \frac{G_1}{s^2}E_{11}(s) + \int_0^s \frac{H_{ij}^1\mathrm{e}^{-2\tilde{k}_j z/s}}{s^3}E_{11}(z)\,\mathrm{d}z F(s) + \frac{G_2}{s^2}E_{33}(s) + \int_0^s \frac{H_{ij}^2\mathrm{e}^{-2\tilde{k}_j z/s}}{s^3}E_{33}(z)\,\mathrm{d}z\end{aligned} \tag{3.21}$$

where

$$\begin{aligned}&G_1 = \alpha_i C_{ij}\mathrm{e}^{-2\tilde{k}_j} \text{ and } H_{ij}^1 = 2\alpha_i C_{ij}\tilde{k}_j;\, G_2 = \beta_i C_{ij}\mathrm{e}^{-2\tilde{k}_j} \text{ and } H_{ij}^2 = 2\beta_i C_{ij}\tilde{k}_j \text{ and} \\ &C_{ij} = -\frac{L_0}{Q}(\pi cn)^2 f_{ij} \text{ as in the previous section.}\end{aligned} \tag{3.22}$$

A numerical approximation of integration in the set of equations (3.21) can considerably reduce the complexity in solving for $\sigma_{11}(z)$ and $\sigma_{33}(z)$. To facilitate this, we rewrite (3.21):

$$\begin{aligned}s^2F^{3(1+3)}(s) &= G_2\left(E_{11}(s) + \frac{1}{G_2}\int_0^s \frac{H_{ij}^2\mathrm{e}^{-2\tilde{k}_j z/s}}{s^3}E_{11}(z)\,\mathrm{d}z\right) + G_1\left(E_{33}(s) + \frac{1}{G_1}\int_0^s \frac{H_{ij}^1\mathrm{e}^{-2\tilde{k}_j z/s}}{s^3}E_{33}(z)\,\mathrm{d}z\right) \\ s^2F^{1(1+3)}(s) &= G_1\left(E_{11}(s) + \frac{1}{G_1}\int_0^s \frac{H_{ij}^1\mathrm{e}^{-2\tilde{k}_j z/s}}{s^3}E_{11}(z)\,\mathrm{d}z\right) + G_2\left(E_{33}(s) + \frac{1}{G_2}\int_0^s \frac{H_{ij}^2\mathrm{e}^{-2\tilde{k}_j z/s}}{s^3}E_{33}(z)\,\mathrm{d}z\right)\end{aligned} \tag{3.23}$$

Now the numerical reduction yields:

$$s^2F^{3(1+3)} = G_2K_2E_{11} + G_1K_1E_{33}$$
$$s^2F^{1(1+3)} = G_1K_1E_{11} + G_2K_2E_{33} \tag{3.24}$$

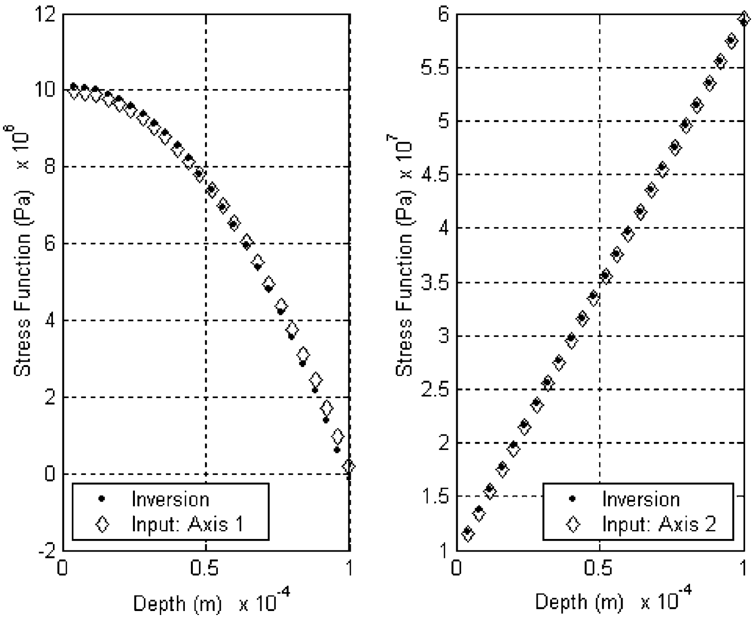

Fig. 5. Simulation result for biaxial stress of form $\sigma = (-A)z^2 + Bz$ along axis 1 and $\sigma = Cz + D$ along axis 2. (A, B, C, D positive.)

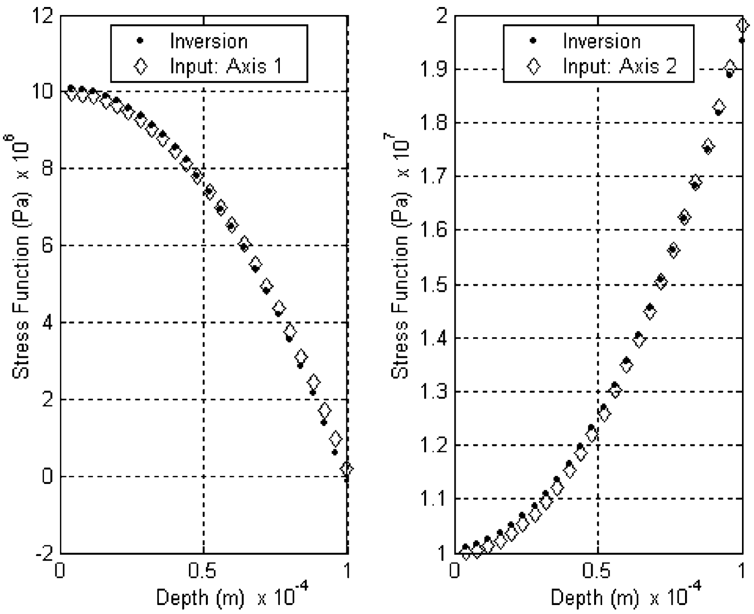

Fig. 6. Simulation result for biaxial stress of form $\sigma = (-A)z^2 + Bz$ along axis 1 and $\sigma = Cz^2 + Dz$ along axis 2. (A, B,C, D positive.)

where $F^{3(1+3)}$ and $F^{1(1+3)}$ are column matrices containing sample values of $F^{3(1+3)}(s)$ and $F^{1(1+3)}(s)$ respectively, E_{11} and E_{33} are column matrices containing sample values of $E_{11}(s)$ and $E_{33}(s)$ respectively, and K_1 and K_2 are coefficient matrices as defined in Eq. (3.19), for specific values of $H_{ij} = H_{ij}^1$ and $H_{ij} = H_{ij}^2$ respectively. (3.24) further yields,

$$\begin{bmatrix} G_2K_2 & G_1K_1 \\ G_1K_1 & G_2K_2 \end{bmatrix} \cdot \begin{bmatrix} E_{11} \\ E_{33} \end{bmatrix} = \begin{bmatrix} s^2F^{3(1+3)} \\ s^2F^{1(1+3)} \end{bmatrix} \tag{3.25}$$

We note that $G_1K_1 \neq G_2K_2$ and therefore we can solve (3.25) successfully to obtain unique solutions E_{11} and E_{33}, and subsequently, the stress functions $\sigma_{11}(z)$ and $\sigma_{33}(z)$ from Eq. (3.9).

3.4. Simulations results

Again, present preliminary simulation results are presented for the case of presence of a biaxial stress state. The numerical procedure adopted and the simulation methodology is same as described in Section 3.2. Fig. 5 shows simulation result for a biaxial stress state where along one axis, we have a quadratic variation of stress with depth and along the other axis we have a linear variation of stress with depth. Fig. 6 shows simulation result for a biaxial stress state where we have a quadratic variation of stress with depth along both the axes. Again, all calculations are in the '*s*' plane.

4. Discussion

4.1. Effect on ill-posedness

The complex physical nature behind the propagation of Rayleigh surface waves manifests itself in the forward model being Fredholm integral equation of the first kind (3.1). This makes the direct inversion of this equation an ill-posed problem. The inversion method proposed in this paper is based upon the conversion of this equation into a Volterra integral equation of the second kind (3.13). It is a known fact in literature (Jerry, 1999) that such equations are less prone to ill-posedness; a mathematical analysis of these equations will reveal the salutary effect of this process.

In Eq. (3.1) the integration over the sought solution, $\sigma(z)$ is a smoothing process. This is further aided by the nicely behaved kernel, $K_{\mathrm{F}}^{\mathrm{I}}(\omega, z)$. Therefore information regarding the variations in $\sigma(z)$ tends to be suppressed in $\delta\phi(\omega)$, the resultant of the integration. Hence, given a function $\delta\phi(\omega)$, we cannot in general, be assured of an answer in search for $\sigma(z)$. This is at the root of how ill-posedness manifests in the Fredholm formulation.

The improvement achieved by the proposed method can be perceived through (3.11). Here, the presence of an additional term containing $E(z)$ (which in effect, contains all the information of $\sigma(z)$) outside the integral operation, ensures the preservation of full information regarding $\sigma(z)$ in $F(s)$. This is how the Volterra formulation is less prone to ill-posedness.

4.2. The choice of a suitable value for '*n*'

It must be noted that, apart from the condition that it must be a finite number, the issue of the value for '*n*' does not figure up to the point where the conversion of integral equation type is effected. Until here, its role is only so far as to make the upper limit of integration dependent upon frequency (see (3.1)–(3.3)) so that a Volterra equation results from a Fredholm equation. But the matrix K which is inverted and applied to the function F to obtain the stress function E in (3.19) is dependent on the value of

'n'. In this matrix 'n' appears twice in the denominator, outside and within a negative exponent; hence the lower the value of 'n' the better, so that the matrix K remains numerically invertible. The range of frequencies used is 176 MHz to 17.6 GHz, corresponding to the depths of 1–100 μm, and were investigated in the simulations. However, this is by-passed during simulations as we are dealing in 's' space, the 'wavelength space'. The wavelength is roughly the penetration depth, so the frequency must be such that the wavelength offers the penetration we require. The Rayleigh wave is known to vanish (Kline et al., 1996; Lindgren et al., 1993) beyond approximately 1.2 times its wavelength so the lowest value that can be chosen for 'n' is 1.2. The value of n between 1 and 1.4 were tested and the inversion was found to be stable.

4.3. Route to the equation of second kind

We could also have arrived at the Volterra equation of the second kind, by an alternative approach. This would have involved a partial differentiation of $F(s)$ in Eq. (3.5) with respect to 's' and usage of the Leibnitz formula,

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_{\alpha(x)}^{\beta(x)} F(x,y)\,\mathrm{d}y = \int_{\alpha(x)}^{\beta(x)} \frac{\partial F(x,y)}{\partial x}\,\mathrm{d}y + F(x,\beta(x))\frac{\mathrm{d}\beta}{\mathrm{d}x} - F(x,\alpha(x))\frac{\mathrm{d}\alpha}{\mathrm{d}x} \tag{4.1}$$

But the method proposed in this work has its advantages. First, the loss of information due to differentiation (whereby a constant term, if present, vanishes) is prevented. Further, the possible magnification of errors in $F(s)$ is avoided.

4.4. Discretization procedure

In this work, a method based upon quadrature rules was used to discretize the integral in Eq. (3.13), to demonstrate the inversion. However, such a procedure is not considered ideal, because of their requirement of a suitable 'starting procedure' and possible propensity to amplification of noise. Self starting block methods such as the Runge–Kutta method (Delves and Mohammed, 1985) are suggested as basis for an alternative.

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Appendix A

The power flow 'P' occurring in Eq. (2.1) is given by:

$$P = \frac{\omega\rho_0 V_0}{2}\left[\frac{(1/V_0^2)+k_s^2}{2k_s} - 2\frac{(K_2/V_0^2)+K_4k_s^2}{k_s+k_l} + \frac{(K_2^2/V_0^2)+K_4^2k_s^2}{2k_l}\right]$$

With the quantities

$$k_s \equiv \sqrt{\left(\frac{1}{V_0^2} - \frac{1}{V_s^2}\right)}; \quad k_l \equiv \sqrt{\left(\frac{1}{V_0^2} - \frac{1}{V_l^2}\right)}; \quad K_2 = \frac{2k_s k_l}{(1/V_0^2) + k_s^2}; \quad K_4 = \frac{2}{(1 + V_0^2 k_s^2)}$$

where V_0, V_s and V_l, the phase velocities of the Rayleigh, transverse and longitudinal waves respectively in the unstressed medium. ρ_0 denotes the constant density of the medium before application of the static initial deformation.

The 15 constants f_{ij}, $i \in \{1,2,3,4,5\}$, $j \in \{1,2,3\}$ occurring in Eq. (2.4) are:

$$f_{11} = (k_s/V_0)^2; \quad f_{12} = (k_l K_2/V_0)^2; \quad f_{12} = 2k_s k_l K_2/V_0^2$$

$$f_{21} = f_{11}; \quad f_{22} = K_4^2 f_{21}; \quad f_{23} = -2K_4 f_{21}$$

$$f_{31} = -2f_{11}; \quad f_{32} = k_l K_2 K_4/k_s; \quad f_{33} = -[K_4 + k_l K_2/k_s] f_{31}$$

$$f_{41} = (1/V_0^4) + k_s^4; \quad f_{42} = K_2^2/V_0^4 + (k_s k_l K_2)^2; \quad f_{43} = -2[K_2/V_0^4 + (k_s^3 k_l K_4)]$$

$$f_{51} = 2f_{11}; \quad f_{52} = k_l K_2 K_4 f_{51}/k_s; \quad f_{53} = -[k_l K_4/k_s + K_2] f_{51}$$

The constants α_i ($i \in \{1,2,3,4,5\}$) appearing in Eq. (2.5) are given by:

$$\alpha_1 = \frac{1}{3\lambda + 2\mu}\left\{\lambda + 2l - \frac{\lambda(2\lambda + 6\mu + 4m)}{2\mu}\right\}$$

$$\alpha_2 = \frac{1}{3\lambda + 2\mu}\left\{\lambda + 2l + \frac{(\lambda + \mu)(2\lambda + 6\mu + 4m)}{\mu}\right\}$$

$$\alpha_3 = \frac{1}{3\lambda + 2\mu}\left\{\lambda + 2l + \frac{\lambda(\lambda + 2m - n)}{2\mu}\right\}$$

$$\alpha_4 = \frac{1}{3\lambda + 2\mu}\left\{3\lambda + 2\mu + m - \frac{\lambda(2\mu - n/2)}{2\mu}\right\}$$

$$\alpha_5 = \frac{1}{3\lambda + 2\mu}\left\{\lambda + \mu + m - \frac{\lambda(\mu - n/2)}{2\mu}\right\}$$

The constants β_i appearing in Eq. (2.6) are:

$$\beta_1 = \beta_2 = \alpha_1$$

$$\beta_3 = \frac{1}{3\lambda + 2\mu}\left\{\lambda + 2l - \frac{(\lambda + \mu)(\lambda + 2m - n)}{\mu}\right\}$$

$$\beta_4 = \beta_5 = \frac{1}{3\lambda + 2\mu}\left\{m - \lambda - \frac{(\lambda + \mu)n}{2\mu}\right\}$$

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